Survival probability in a one-dimensional quantum walk on a trapped lattice

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Abstract. The dynamics of the survival probability of quantum walkers on a one-dimensional lattice with random distribution of absorbing immobile traps are investigated. The survival probability of quantum walkers is compared with that of classical walkers. It is shown that the time dependence of survival probability of quantum walkers has a piecewise stretched exponential character depending on the density of traps in numerical and analytical observations. The crossover between the quantum analogs of the Rosenstock and Donsker-Varadhan behaviors is identified.

PACS numbers: 05.40.Fb, 03.67.Lx, 03.65.Yz

1. Introduction

The classical random walk (CRW) is a prototype model of stochastic processes that occur in many physical systems [1]. Extension of random walk concept from stochastic classical realm to the unitarily evolving quantum world is motivated by the promise of quantum walks (QWs) [2, 3] as quantum algorithms [4] outperforming their classical counterparts, and as a simple model for quantum computation.

In parallel with the remarkable developments in the experimental ability to control single atoms and photons, early proposals and demonstrations of QWs are followed by more robust and controllable implementations [5, 6, 7]. Superiority of a QW algorithm [4] is experimentally demonstrated very recently [8]. In a latest experiment [9], the effect of absorbing boundaries on the quantum walk is examined. The QW on a line segment with absorbing boundaries [2, 10] is a special case of a more general situation of QW in the presence of absorbing traps.

A QW on a trapped lattice exhibits transition to CRW [11]; therefore traps can be characterized as a quantum decoherence mechanism [12, 13], similar to broken links [14] or environmental noise [15]. Controlling trap density in the lattice allows for tunable decoherence mechanism, which is beneficial for fundamental investigations of quantum decoherence and for efficient implementation and speeding up quantum algorithms [16, 17].

In addition to their role as a source of quantum decoherence, traps can play another role on the dynamics of QW. It is known that their presence causes different dynamical regimes on the evolution of CRW. Our aim is to explore if such distinct dynamical regimes can emerge in QW without changing its quantum nature. Earlier occurrences of such a crossover between different dynamical regimes than quantum to classical transition should be taken into account potential applications of trapped QW.

Trapped CRW was extensively explored [18, 19, 20]. A practical quantity of interest is the survival probability of diffusing particles, which is the mean probability that a walker can still be found on the lattice after some time t. It can be analytically calculated for a one-dimensional CRW [21]. In the early times of CRW on a one-dimensional lattice with low trap concentration, survival probability decays exponentially with the square root of time, $t^{1/2}$, which is known as Rosenstock (RS) [22] behavior. At asymptotically large times, this behavior makes a crossover [23] to a qualitatively different scaling form, which is called Donkser and Varadhan (DV) regime [24, 25, 26, 27], in which the survival probability exponentially decays with $t^{1/3}$. Similar scaling forms also appear in the closely related problem of Lifshitz tail or Griffiths singularity of the density of states at the band edge for a quantum electron in random potential [28, 29, 30].

It is neither intuitively nor quantitatively obvious to extend the classical results to characterize the survival probability of quantum walkers on a trapped land, because of the curious role of quantum coherence and path interference played in a QW which is associated with the characteristic strong de-localization of quantum walkers. This paper specifically addresses the question of quantum diffusion dynamics on a trapped chain,

in particular investigates the quantum analogs of the RS and DV dynamical regimes of the classical diffusion problem.

It is difficult to observe the RS-to-DV crossover in CRW. Time dependence of classical survival probability makes the crossover only after a long time, though it can happen relatively earlier for larger trap concentrations or for larger diffusive constants [31]. Motivated by the role of the diffusive constant in shortening the crossover time in CRW, we predict that highly de-localized quantum walkers can enter the DV regime earlier than the classical walkers. Furthermore, the qualitatively different DV scaling form in CRW is attributed to the existence of large voids, or absorber-free regions, which are exponentially rare among the possible configurations [26]. Their contribution can dominate the time dependence of the survival probability only at large times, after the more common smaller voids lost their walkers. In QW, we expect that, due to the larger spread of the walkers, such voids should be larger, and hence more rare. An average over such clusters, with their corresponding large decay times, would lead to slower decay of quantum walkers than their classical counterparts. Indeed, for a continuous time one-dimensional quantum transport problem, it is found that the survival probability exponentially decays with $t^{1/4}$ [32]. This asymptotically slower decay of quantum coherent particles than the diffusive classical particles is explained by the existence of slowly-decaying asymptotically large trap-free segments [32]. In QW-based search algorithms with multi-agents, such slowing down of quantum coherent dynamics would cause an additional limitation of the quantum speeding up even for small target (or trap) concentrations. It should be taken into account in addition to the usual quantum-to-classical decoherence problem.

We perform detailed numerical simulations for settings relevant to current experimental efforts. As such, our discussion is limited to the one-dimensional coined discrete-time QW. Comparative studies of signatures of coherent and incoherent transport in the case of continuous-time quantum walk are recently reported [32, 33, 34]. We give particular attention to the practical case of small, finite size lattices and small number of time steps. Similar to the classical prototype system of disordered media, randomly distributed static traps are assumed. The trapping process is supposed to be a quenched, instantaneous and perfect absorption of walkers. In a typical scenario of interest there would be few traps, and dynamics would be limited to short times; but the cases of long time behavior as well as densely trapped lattices are also analyzed to comprehend differences in both RS and DV regimes, in addition to dynamics of quantum-to-classical transition.

We explain our numerical results by the Flory-type heuristic arguments [35] used in polymer chemistry. Spatial arrangements of macromolecules, or conformations of polymers, are closely related to the diffusion and the random walk problem. In the early 1930s, structural chemistry descriptions of long-chain molecules were based upon unconstrained random walks, where the skeletal bonds of the molecule are represented by the uncorrelated steps of random walkers. This analogy yields scaling relations for the root mean square (rms) distance of the chain (squared radius of gyration) depending

on the bond length and the number of the bonds. In 1949, P. J. Florry has provided a seminal work which takes into account volume exclusion effect (no segments of a molecule can overlap in space), in formation of polymers. This allows for description of polymer growth in terms of the self-avoiding or repulsive random walks. In self-avoiding walks, the walker would stop or become trapped if there are no more unvisited neighboring sites. Our trapped lattice model is in that sense is closely related to such random walk models of polymer growth and size distribution. The movement of a single walker to a nearest neighbor site can be imagined as initiating formation of an unsaturated bifunctional monomer, while trapping would give polymers of different sizes.

This paper is organized as follows. We first present a short review of the theory and major results of the one-dimensional QW in Sect. 2. The model of CRW and QW with traps is introduced and the survival probability is defined in Sect. 3. In Sect. 4, the numerical simulations on the survival probability are shown and the analytical results are given using the correspondence to the QW with position measurement on the line in the thermodynamic limit. Section 5 is devoted to the conclusions and the outlooks.

2. Quantum walk

2.1. Single-particle walk

We consider a coined discrete-time QW on a finite linear lattice segment with periodic boundary conditions. Denoting the total number of sites on the lattice by K, the geometry is equivalent to a ring, or a so-called K-cycle [16, 36]. In strict mathematical terms, it is the Cayley graph of the cyclic group of size K. The coin (chirality) space of a single walker is described by $\mathcal{H}_C$ with two basis vectors $\{|\uparrow\rangle, |\downarrow\rangle\}$. Also, the position space of a single walker on this chain is described by $\mathcal{H}_P$ with the basis $\{|k\rangle \ : \ k \in \mathbb{Z}/K\mathbb{Z}\}$. The Hilbert space of total system is given by $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$. We identify the chirality basis vectors as

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{1}$$

Each step of the particle ‡ consists of a unitary coin operation $\hat{C}$ for the chirality transformation, and a position-shift operation $\hat{S}$. At time t, QW is defined by transformation $\hat{U}^t$ with $\hat{U}$ being the unitary operator of a walk step which is given by

$$\hat{U} := \hat{S}(\hat{C} \otimes \hat{I}), \tag{2}$$

with $\hat{I}$ being the identity operator. Throughout this paper, we assume that the coin operator $\hat{C}$ is the Hadamard operator

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3}$$

‡ Throughout this paper, we call it the step or the time. It should be noted that these have the same meaning.

for simplifying the discussion. The shift position operator $\hat{S}$ is described by

$$\hat{S} = |\uparrow\rangle\langle\uparrow| \otimes \sum_{k=1}^{K} |k+1\rangle\langle k| + |\downarrow\rangle\langle\downarrow| \otimes \sum_{k=1}^{K} |k-1\rangle\langle k| \,, \tag{4}$$

with $k \in \mathbb{Z}/K\mathbb{Z}$; $K+1 \equiv 1$ and $0 \equiv K$. The wave function of quantum walker at time t can be written as $|\psi(t)\rangle = \sum_{c,k} \psi_c(k,t) \, |c,k\rangle$ with $c = \uparrow, \downarrow$. This can be rewritten as

$$|\psi(k,t)\rangle = \sum_{c\in\{\uparrow,\downarrow\}} \psi_c(k,t) \, |c\rangle = \begin{bmatrix} \psi_\uparrow(k,t) \\ \psi_\downarrow(k,t) \end{bmatrix}, \tag{5}$$

where $\psi_\uparrow(k,t)$ and $\psi_\downarrow(k,t)$ represent probability amplitudes of the particle at the site k at time t, depending upon the internal states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

At time t, the quantum state of the quantum walker is given by

$$|\psi(t)\rangle = \hat{U}^t \, |\psi(0)\rangle \,, \tag{6}$$

where $|\psi(0)\rangle = |\chi, m\rangle$ is the initial state of the coin $|\chi\rangle$ and the position $|m\rangle$ $(m \in \mathbb{Z}/K\mathbb{Z})$. Here, we define the density operator of the quantum walker as $\Phi(t) = |\psi(t)\rangle \langle\psi(t)|$. Then, the probability distribution of walker at position x at time t can be calculated by

$$P(x,t) = \sum_{c\in\{\uparrow,\downarrow\}} \langle c,x| \, \Phi(t) \, |c,x\rangle \quad (x \in \mathbb{Z}/K\mathbb{Z}). \tag{7}$$

This can be rewritten as

$$P(x,t) = |\psi_\uparrow(x,t)|^2 + |\psi_\downarrow(x,t)|^2 \,. \tag{8}$$

2.2. Multi-particle walk

One can easily envision that multi-particle random walks can be more advantageous in search algorithms than single-particle ones. Indeed, recent experimental progress and theoretical studies favor the many-body random walk problem both in classical and in quantum realms [37, 38]. As the general approach in terms of indistinguishable, correlated and interacting particles to this problem is too challenging to start with, we aim to comprehend the simplest scenario in this work and consider the complications in particular implementation settings in future studies. Let us assume the walkers are non-interacting distinguishable particles and they are initially uncorrelated. For N such walkers, the Hilbert space is given by a direct product of single walker spaces,

$$\mathcal{H} = \bigotimes_{i=1}^{N} (\mathcal{H}_C \otimes \mathcal{H}_P)_i, \tag{9}$$

with the particle label i. The particles walk independent of each other on the K-cycle so that the time evolution of the whole system is determined by

$$\hat{U}_{1,2,\ldots,N} := \hat{U}^{\otimes N} \tag{10}$$

where $\hat{U}$ is given by Eq. (2) and is the same for all particles.

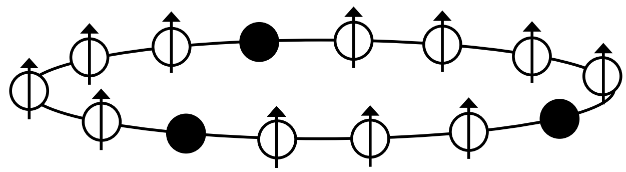


Figure 1. Schematic representation of multi-particle QW on the K-cycle for 11 walkers with the initial state $|\uparrow\rangle$ and 3 absorbing trapped sites (black color).

The initial state of N walkers is given by a tensor product of the single walker initial states as

$$|\Psi(0)\rangle = \bigotimes_{i=1}^{N} |\chi, m_i\rangle_i \,, \tag{11}$$

where $|\chi, m_i\rangle_i$ expresses the ith particle state with the chirality $|\chi\rangle$ and the position $m_i = 1, \ldots, K$. This is illustrated in Fig. 1. At time t, the quantum state of the system becomes

$$|\Psi(t)\rangle = \hat{U}^t_{1,2,\ldots,N} \, |\Psi(0)\rangle \,. \tag{12}$$

Using the reduced single-particle density matrix

$$\Phi_i(t) = \mathrm{Tr}_{j\neq i} \, |\Psi(t)\rangle \langle\Psi(t)| \,, \tag{13}$$

probability distribution of single walker at time t can be evaluated by

$$P_i(x,t) = \sum_{c\in\{\uparrow,\downarrow\}} \langle c, x| \, \Phi_i(t) \, |c, x\rangle \quad (x \in \mathbb{Z}/K\mathbb{Z}). \tag{14}$$

This shows that $P_i(x,t)$ can be interpreted as a conditional probability to find a walker at site $x \in \mathbb{Z}/K\mathbb{Z}$ at time t when the particle started to walk from site m_i at $t = 0$. The complete set of $\{P_i(x,t)\}$ for all particles $i = 1, \ldots, N$ can be visualized as the set of transition probabilities from $\{m_i\}$ to x of a single particle, so that the simplest multi-particle QW problem under study here is essentially a single particle problem that starts to walk at a set of different initial locations.

3. Survival probability

We use the exact enumeration method for calculating the survival probability in CRW [25], which is suitable for the randomly distributed immobile traps on a one-dimensional lattice. Initially, every untrapped site is occupied by a walker. At each step, N walkers perform CRW on the one-dimensional lattice, for which the probability of finding a walker at a particular site $P_i(x,t)$ is calculated with the sum of the corresponding probabilities at its nearest neighbor sites divided by two. The survival probability at time t is given by

$$P_r(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{x=1}^{K} P_i(x,t). \tag{15}$$

Here r enumerates a particular independent initial configuration of the system. Note that we have the relation $N = K - n$ or $N = K(1 - \rho)$, where n is the number of traps on the lattice and $\rho = n/K$ is the concentration from our assumption. We take the lattice sites at $\{x : x = 1, ..., K\}$. Note that we assume non-overlapping, immobile, perfectly absorbing sites such that $P_i(x, t) = 0$ if the site x is a trapping site; hence the sum is not restricted to the untrapped sites. Furthermore, in this paper, the initial configurations on the particles are assumed that a single particle is only in each untrapped site.

To account for random distribution of the traps, a statistical configurational average of mean survival probability is calculated over different independent realizations of the initial system via

$$\langle P(t) \rangle = \frac{1}{M} \sum_{r=1}^{M} P_r(t), \tag{16}$$

where M denotes the number of different configurations.

Let us now examine the quantum analog of the survival probability [32, 39, 41]. In QW, quantum states of the particles lead to non-trivial path interference effects. The quantum states can be initialized arbitrarily before the walk starts. At each time step, new positions and states of quantum walkers are determined with the unitary transformation $\hat{U}$ for each walker. If a state meets with an immobile trap, it gets annihilated. In the previous section we have seen that for our simplified case, the process is equivalent to a single-particle problem with an ensemble of initial configurations. Thus we can use the classical definition of the survival probability by only making a quantum mechanical calculation of the single-particle probability distribution.

Let us briefly consider the experimental realization of this system. In the system of the two-dimensional ion trap experiment [5], it seems to be possible to manipulate the QW on the ring. Localized ion losses can be effectively considered as the absorption traps. Also, in the system using the photon by the waveguide, the non-linear phase gate is essentially used to realize the QW on the circle [42]. Combining the waveguides, the absorption traps could to be realized.

4. Results and discussions

4.1. Survival probability in QW on a finite one-dimensional trapped lattice

We numerically analyze the dynamics of the survival probability in QW for three different initializations of the system. In the first case, all quantum states at the untrapped sites are initialized as $|\uparrow\rangle$. In the second case, the initial states are randomly assigned either $|\uparrow\rangle$ or $|\downarrow\rangle$. In the last case, all states are chosen as superpositions $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle)$. We shall respectively call them as up, mixed and symmetric initializations.

Typical simulation results are reported in Figs. 2(a)–(c), for the three cases of initialization of the QW, and for different trap densities, $\rho = 0.05$, $\rho = 0.1$, $\rho = 0.2$, and $\rho = 0.3$. The figures are plotted in a convenient double logarithmic scale. While the survival probability exhibits the expected behaviors of decrease in time and being

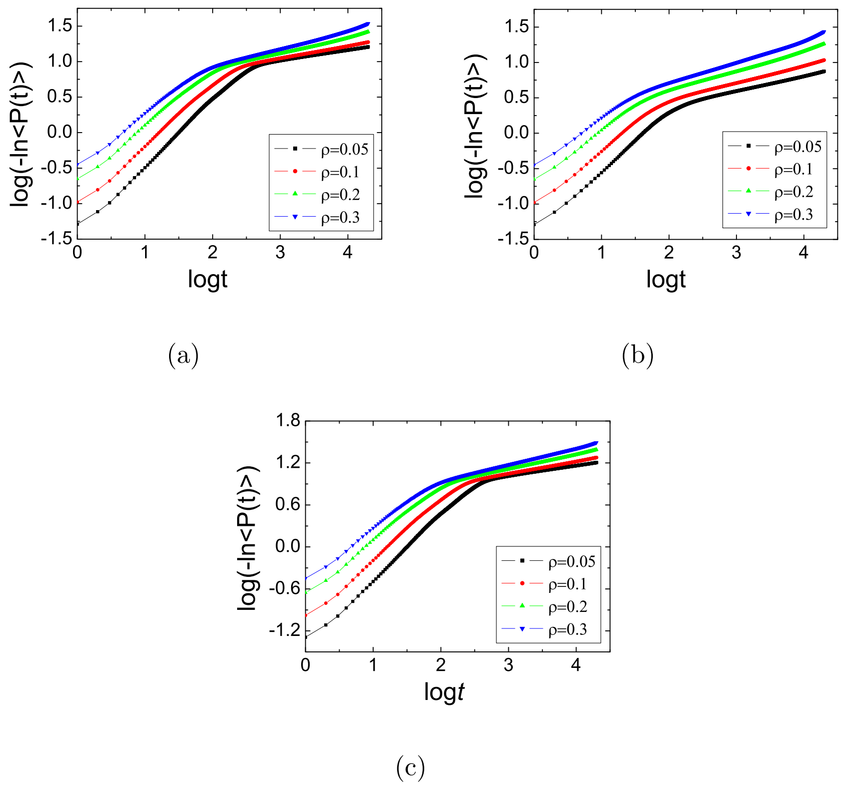


Figure 2. (a) The time dependence of the survival probability in the QW on a lattice of $K = 101$ sites for $T = 20000$, $M = 10000$, $\rho = 0.05$, $\rho = 0.1$, $\rho = 0.2$ and $\rho = 0.3$ with the initializations a) $|\uparrow\rangle$ (up), (b) randomly distributed $|\uparrow\rangle$ or $|\downarrow\rangle$ (mixed), and (c) $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle)$(symmetric).

less at higher trap densities, it invites a closer look due to some non-trivial qualitative changes in its dynamics. All the different initializations lead to two qualitatively different dynamical regimes of survival probability. These two regimes makes a crossover at a certain time point, t_c, whose location depends on the trap density. The crossover time t_c appears at $t_c \approx 25/\rho$ for up and symmetric initializations, and at $t_c \approx 8/\rho$ for mixed initialization. Figure 2 is plotted for $K = 101$ sites but we also tried different lattice sizes and found similar results to Fig. 2.

As seen in Fig. 3, the "mixed" initial configuration behaves dynamically different than "up" and "symmetric" cases. This is due to the profound quantum character of pure states in contrast to the statistical mixture, evolving more closer to classical walk. Pure quantum states benefit the fast spread of the quantum walk in the Rosenstock regime more than a statistical mixture can do. Classical walks perform worst in this regime as we shall argue more below.

Before and after the crossover point, the survival probability exhibits a linear dependence on time in the double logarithmic scale. By increasing M the linear behavior becomes more evident including the end and the beginning times. After that it is simple to make linear fits to the curves in the regimes from $t = 1$ to $t = t_c$ and from $t = t_c$ to

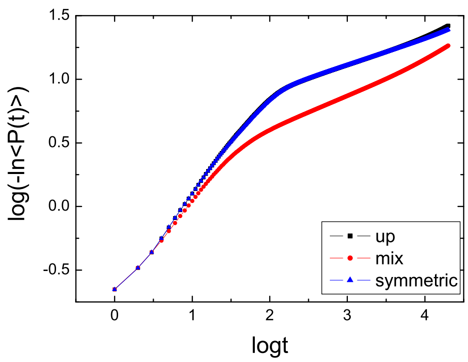


Figure 3. The time dependence of the survival probability in the QW on a lattice of $K = 101$ sites for $T = 20000$, $M = 10000$, $\rho = 0.2$ with the initializations $|\uparrow\rangle$ (up), randomly distributed $|\uparrow\rangle$ or $|\downarrow\rangle$ (mixed) and $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle)$ (symmetric).

$t = T$, the end of the walk. The line fits yield the Kohlrausch-Williams-Watts stretched exponential function [40] description of the survival probabilities, given by

$$\langle P\left(t\right)\rangle \sim \exp\left[-t^{\beta}\right], \tag{17}$$

where the stretching exponent $0 < \beta < 1$ determines the decay rate of $\langle P\left(t\right)\rangle$. It gets different values, β_1 and β_2, before and after t_c, respectively. Their dependence on trap density ρ is shown in Fig. 4. β_1 decreases monotonically with the increasing ρ, whereas β_2 increases with it.

The value of β_1 at low ρ can be understood following the classical RS approximation method. When s_t is the number of distinct sites visited at time t, the probability of being not absorbed for a single walker can be written as $p_t = (1-\rho)^{s_t}$. Formally the mean probability can be expressed in the form $P(t) = \langle p_t\rangle = \langle \exp\left(-\lambda s_t\right)\rangle$ with $\lambda = -\ln\left(1-\rho\right)$. Employing the RS approximation for short time and small ρ, we get $P(t) \approx \exp\left(-\lambda\langle s_t\rangle\right)$. For the CRW, $\langle s_t\rangle \sim \sqrt{t}$ gives the usual RS scaling form. For the QW, however, the ballistic spread up of the quantum walkers allows for $\langle s_t\rangle \sim t$ [43, 44] so that the equivalent rate of survival is enhanced to $\beta_1 \sim 1$ in the quantum analog of the RS regime.

The maximal value of $\langle s_t\rangle$ is associated with the screening or penetration length that measures the distance between the starting and the trapping site of the walkers. Quantum walkers have larger penetration lengths than classical ones and as such can survive at asymptotic times provided that they start in larger clusters. The probability of such configurations are exponentially rare with the size of the segment while the quantum spreading is a power law (quadratic) gain relative to classical walk. So it is necessary to find indeed large voids (relatively larger than their classical counterparts) to ensure quantum walkers can survive. In the quantum analog of the DV regime, the size of the dominating, remaining clusters with quantum walkers, therefore would be larger than the classical DV regime. Averaging over such slowly decaying large voids would

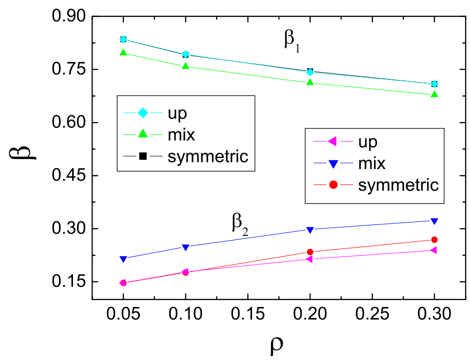


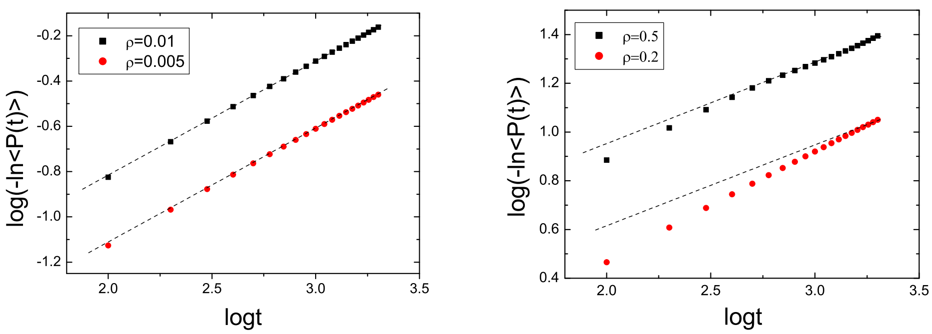
Figure 4. Dependence of the decay parameters $\beta_{1,2}$ on the low trap density ρ in a lattice of $K = 101$ sites, in cases of the up, mixed and symmetric initializations.

then lead to the survival probability decaying slower than the DV regime of classical diffusion. Similar observation for the case of continuous-time quantum transport gives $-\ln\langle P(t)\rangle = t^{1/4}$ [32]. In our case, β_2 is close to this value up to $\rho < 0.3$ as demonstrated in Fig. 4.

The Flory-type heuristic arguments for β_2 and RS approximation for β_1 justify that the numerically observed crossover in Fig. 2 is indeed the quantum analog of classical RS-to-DV transition. It is known that such a crossover can happen only at long times and hard to observe in the CRW. In classical systems, t_c can be reduced either by increasing ρ, which is especially efficient for one dimension [31], or by using systems with large diffusive constants [31]. Remarkably, decrease of t_c with ρ is also observed in Fig. 2 for QW. Increase of ρ however would mean to loose the benefits of the quantum coherence in QW due to quantum-to-classical transition that happens at high ρ [11, 14, 15]. On the other hand, strong de-localization of quantum walkers contributes significantly for further reduction of t_c in the QW. As such, we expect that the quantum analog of RS-to-DV crossover can occur earlier than CRW. As the decay of survival probability of QW is even slower than classical diffusion in quantum DV regime, this makes the quantum RS-to-DV crossover a serious limitation to implement QW-based quantum search algorithms even for relatively small number of traps (or targets). The usual limitation factor of quantum-to-classical transition is an issue only for high ρ. In the next section, we shall verify our predictions and also investigate the effect of large trap concentrations. Furthermore, we will analytically show the crossover in the thermodynamic limit.

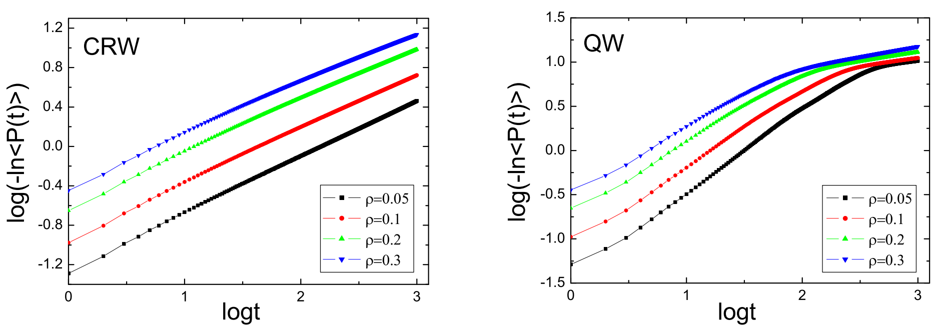
4.2. Survival probability in CRW vs QW

Short and long time behaviors of the survival probability in the CRW and evidence of RS-to-DV crossover are shown in Fig. 5. In short time behavior, the survival probability at low trap densities comply with RS behavior and fit well to $\beta = 1/2$ in Fig. 5(a). If



(a) (b)

Figure 5. Time dependence of the survival probability in CRW on a lattice of $K = 50000$ sites for (a) $T = 2000$, $M = 100$ and $\rho = 0.01$, $\rho = 0.005$, where the broken lines represent the slope of $\beta = 1/2$, (b) $T = 2000$, $M = 100$ and $\rho = 0.2$, $\rho = 0.5$ where the broken lines represent the slope of $\beta = 1/3$.



(a) (b)

Figure 6. The time dependence of the survival probability for a lattice of $K = 101$ sites, $T = 1000$, $M = 100000$ (a) in the CRW, (b) in the QW with the initial state $|\uparrow\rangle$.

the trap density is increased, the slope decreases and approaches 1/3 for large values of ρ. DV behavior emerges in Fig. 5(b). These analytical values are strictly valid for thermodynamically large system. Convergence to the asymptotic DV scaling form is faster in case of higher trap concentrations.

To compare the CRW and QW, we consider a lattice of $K = 101$ sites and take $T = 1000$. The time dependence of survival probability is shown in Fig. 6(a) for the CRW and Fig. 6(b) for the QW. We choose the initialization that gives the longest t_c, to consider the worst situation for the QW. Even for this case, we see that quantum RS-to-DV crossover happens while the CRW is still in the classical RS regime. In particular for low ρ, highly distinct and clear crossover can be observed in the QW.

From the plots on survival probability in Fig. 6, the influence of ρ on the scaling forms can be systematically investigated by Fig. 7, in which the stretching exponents

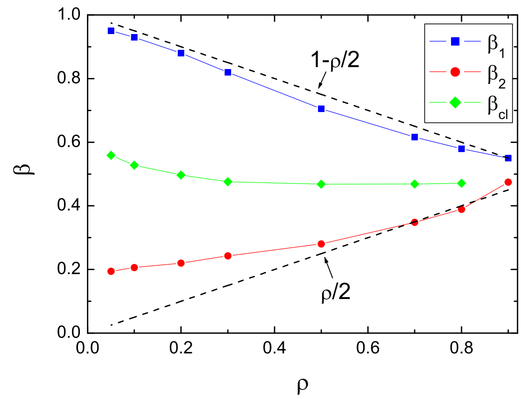


Figure 7. Dependence of the decay parameter β on the trap density ρ for a lattice of $K = 101$ sites and $t = 1000$ time step in the CRW and the QW with initial state $|\uparrow\rangle$. Dashed lines represent analytical fitting results.

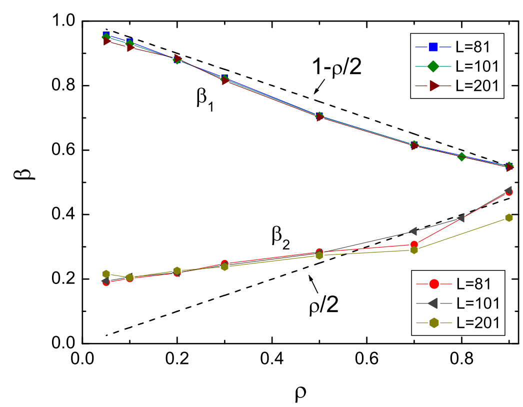


Figure 8. Dependence of the decay parameter β on the trap density ρ for lattice of $K = 81$, $K = 101$ and $K = 201$ sites, $t = 1000$ time step in the QW with initial state $|\uparrow\rangle$.

are plotted as functions of trap density ρ. As the trap concentration increases, β_1 and β_{cl} decrease. In contrast, β_2 increases with ρ. As noted earlier, the sharpest transition between quantum RS and DV regimes happens at low ρ. In high trap densities, their separation shrinks and both $\beta_{1,2}$ converges to the classical exponent β_{cl}. This is in accordance with the expectation that for such high ρ, decoherence transition of QW to CRW should occur. Figure 7 gives strong and clear evidence that the dynamical transition in Fig. 6(b), between early and longer time scaling forms of QW, is not a quantum-to-classical transition, but the true quantum analog of classical RS-to-DV crossover.

4.3. Single-particle QW with position measurement and survival probability in multi-particle QW

In this subsection, we analytically show the relationship between the survival probability and the QW with position measurement on the line.

Let us recapitulate the QW with position measurement on the line [13, 45, 46]. Here, we replace the position Hilbert space to $\tilde{\mathcal{H}}_P = \{|z\rangle : z \in \mathbb{Z}\}$. The one-step dynamics is given by

$$\begin{aligned}\Phi(t+1) &= (1-p)U\Phi(t)U^\dagger \\ &+ p\,|\chi\rangle\langle\chi| \otimes \left[\mathrm{Tr}_C \sum_{z,z' \in \mathbb{Z}} \left[(\hat{I} \otimes |z\rangle\langle z|)\hat{U}\Phi(t)\hat{U}^\dagger(\hat{I} \otimes |z'\rangle\langle z'|)\right]\right],\end{aligned} \tag{18}$$

where $\Phi(0) = \left|\tilde{\psi}(0)\right\rangle\left\langle\tilde{\psi}(0)\right|$ with $\left|\tilde{\psi}(0)\right\rangle = |\chi, 0\rangle$ and $p \in [0, 1]$. This model can be taken as the position measurement of the one-dimensional QW with probability p. When $p = 1/t^\gamma$ $(0 \leq \gamma \leq 1)$, the asymptotic behavior of the QW with position measurement is $\langle s_t^{(D)}\rangle \sim t^{(1+\gamma)/2}$ [13, 45].

In the case of the thermodynamic limit, $K \to \infty$ with fixed ρ, and the sufficiently large t, many-particle QW on the K-cycle can be reduced to the single-particle QW on the line as follows. For uncorrelated quantum walkers and quantum coin in our model of multi-particle QW, the event of annihilation of the walker reaching a trap site is equivalent to a position measurement. The mean probability that the particle reaches the trapped site at time t is $p = t^\rho/t = (1/t)^{(1-\rho)}$. Therefore, it is possible to apply the result on the QW with position measurement on the line to this system to obtain the asymptotic behavior of the QW to arrive at the trapped site as $\langle s_t^{(T)}\rangle \sim t^{1-\frac{\rho}{2}}$. This can be taken the mean free path. In the thermodynamic limit, we can apply the central limit theorem to obtain that the survival probability is the exponential decay for the mean free path as

$$\langle P(t)\rangle \sim \exp\left[-\frac{\langle s_t^{(NT)}\rangle}{\langle s_t^{(T)}\rangle}\right] \sim \exp\left[-t^{\frac{\rho}{2}}\right], \tag{19}$$

where $\langle s_t^{(NT)}\rangle \sim t$ is the mean free path without the trap site, *i.e.*, the QW behavior without position measurement.

Let us reconsider the thermodynamic limit with the fixed ρ for the two types: $t \ll K$ and $t \sim K$. In the first case, *i.e.*, before the crossover time, the survival probability can be approximately taken as the small t. That is, it is impossible to directly apply Eq. (19). Analogous to the discussion in the above section, the survival probability can be rewritten as

$$\langle P(t)\rangle \approx 1 - \langle s_t^{(T)}\rangle \sim 1 - t^{\left(1-\frac{\rho}{2}\right)} \tag{20}$$

From $\langle P(t)\rangle \sim \exp\left[-t^{\beta_1}\right] \approx 1 - t^{\beta_1}$, we obtain

$$\beta_1 = 1 - \frac{\rho}{2} \tag{21}$$

In the second case,*i.e.*, after the crossover time, on the other hand, Eq. (19) can express the exponential decay. Therefore, we directly obtain

$$\beta_2 = \frac{\rho}{2} \tag{22}$$

These analytical results can be compared with the numerical results of Fig. 7. While these show good agreement, we cannot see the finite-size effects as seen in Fig. 8. Also, our numerical calculation is used in the same and specific chirality state for the multi-particle QW and the Hadamard coin. When we remove these conditions, our analytical observation is unchanged in the thermodynamic limit since the essential part of the proof the limit distribution is only the identical distribution for the single particle [45].

5. Conclusion and Outlook

We have investigated the time dependence of the survival probability in discrete Hadamard QW on a K-cycle with random, static, perfect traps. We have found that the survival probability exhibits a piecewise stretched exponential character. In the early time regime, it decays faster than that of CRW, while in the late time regime it decays slower. The crossover time between two regimes decreases with trap density ρ. By analytical and heuristic arguments, we have identified the dynamical transition between two regimes as quantum analog of the RS-to-DV crossover in classical diffusion. We have shown that quantum RS-to-DV crossover can happen earlier than its classical counterpart. At high trap concentrations quantum-to-classical transition happens. At low trap concentrations, even if quantum-to-classical transition does not play a role, quantum RS-to-DV crossover has found to be a serious limitation on the benefits of quantum coherence, such as quadratic speeding up in implementations of QW-based quantum search algorithms.

As an outlook of present work, consideration of larger dimensional systems with probabilistic or state-dependent traps and interacting walkers could make the results more suitable for applications and experimental realizations [47]. From a more fundamental point of view, further investigations of the quantum RS-DV scaling transitions can be performed for in terms of quantum Zeno effect in QW [48] or in relation to random quenched disorder [49]. A more direct and rigorous generalization of Flory mean field theory to the trapped quantum random walk can also be pursued. An extension of our work for the question of quantum RS-to-DV transition in the case of continuous QW would be of interest in the light of the recent realizations of multi-agent continuous QW [50].

Acknowledgments

The authors thank Zafer Gedik, Hamza Polat, Ismail Hakki Duru, Kaan Guven and Jun-ichi Inoue for many discussions and helpful comments. Two of the authors (EA and ÖEM) acknowledge warm hospitality and support by the Dokuz Eylül University.

This work was supported by the TUBITAK (The Scientific and Technological Research Council of Turkey) under research project (No.109T681), İstanbul University (No.3660 and No.6942), and JSPS Research Fellowships for Young Scientists (No.21008624). OEM gratefully acknowledges the support by the DPT (T.C. Prime Ministry State Planning Organization) under the project of National Quantum Cryptology Center for UEKAE (National Research Institute of Electronics and Cryptology). YS is supported by Global Center of Excellence Program "Nanoscience and Quantum Physics" at Tokyo Institute of Technology.

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